

Assume the following 5-year term structure of interest rates:

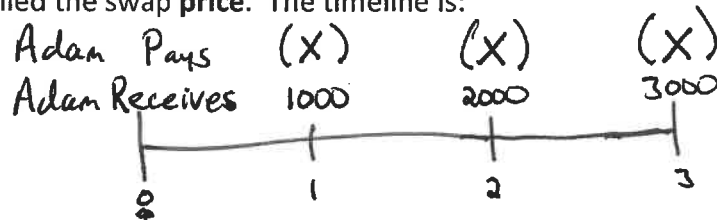
$$\begin{aligned}s_1 &= 0.01 \text{ (1-year spot rate)} \\s_2 &= 0.02 \text{ (2-year spot rate)} \\s_3 &= 0.03 \text{ (3-year spot rate)} \\s_4 &= 0.04 \text{ (4-year spot rate)} \\s_5 &= 0.05 \text{ (5-year spot rate)}\end{aligned}$$

Notation: Define $v_k = \frac{1}{1+s_k}$ and so $(v_k)^k$ is the periodic discount factor from time k to time 0. Even though there is no "0"-year spot rate, for later use define $(v_0)^0 = 1$.

A **swap** is a contract that exchanges one set of payments for another set of payments such that the present values of the two sets of payments are equal using the current term structure of interest rates.

Example: Adam enters into a swap in which payments of 1000, 2000, and 3000, are due at the end of years 1, 2, and 3, respectively. Adam will pay the **counterparty** of the swap a fixed amount X at the end of years 1, 2, and 3, and receive the above non-level payments from the counterparty. Determine X , and the **net swap payment** for year 2.

Solution: This swap has a 3-year **term** (or **tenor**) with annual **settlements**. The value of X is called the swap **price**. The timeline is:



Setting the present values equal to each other, we get

$$X \cdot (v_1) + X \cdot (v_2)^2 + X \cdot (v_3)^3 = 1000 \cdot (v_1) + 2000 \cdot (v_2)^2 + 3000 \cdot (v_3)^3$$

and so

$$X = \frac{1000 \cdot (v_1) + 2000 \cdot (v_2)^2 + 3000 \cdot (v_3)^3}{(v_1) + (v_2)^2 + (v_3)^3}$$

Since $v_1 = \frac{1}{1+s_1} = \frac{1}{1.01}$ and $v_2 = \frac{1}{1+s_2} = \frac{1}{1.02}$ and $v_3 = \frac{1}{1+s_3} = \frac{1}{1.03}$, then $X = 1974$.

For year 2, Adam will receive from the counterparty 2000 and pay the counterparty 1974. Therefore, the net swap payment is $\Delta = 2000 - 1974 = 26$. Adam receives a net payment of 26 from the counterparty.

An **interest rate swap** is a swap in which the payments in the swap are interest payments on a loan. The principle at the beginning of each year is called the **notional** amount of the swap. (Notional means “in name only”.) Letting N_0, N_1, \dots denote the notional amounts at the beginning of year 1, year 2, ..., we get

$N_0 \cdot f_{[0,1]}$ = the amount of interest due at time 1

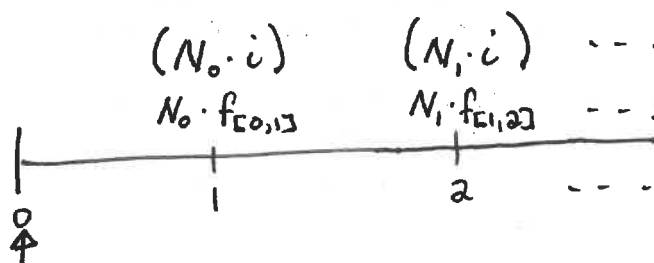
$N_1 \cdot f_{[1,2]}$ = the amount of interest due at time 2, etc.

An interest rate swap replaces these payments with similar payments using a fixed periodic effective interest rate, i , called the **swap rate**. That is,

$N_0 \cdot i$ = the swap amount at time 1

$N_1 \cdot i$ = the swap amount at time 2, etc.

The timeline is:



Setting the two present values equal to each other, we get

$$N_0 \cdot i \cdot (v_1) + N_1 \cdot i \cdot (v_2)^2 + \dots = N_0 \cdot f_{[0,1]} \cdot (v_1) + N_1 \cdot f_{[1,2]} \cdot (v_2)^2 + \dots$$

and solving for the swap rate i , we get

$$i = \frac{N_0 \cdot f_{[0,1]} \cdot (v_1) + N_1 \cdot f_{[1,2]} \cdot (v_2)^2 + \dots}{N_0 \cdot (v_1) + N_1 \cdot (v_2)^2 + \dots}$$

Remarks:

1. In most cases the settlement period is years and the **settlement date** is at the end of the year.
2. The **payer** of the swap is the party that agrees to pay the fixed interest rate and receives the variable interest rate. That is, for period k , the payer receives from the counterparty $N_{k-1} \cdot f_{[k-1,k]}$ and pays the counterparty $N_{k-1} \cdot i$. Therefore, the net swap payment for period k is $\Delta = N_{k-1} \cdot (f_{[k-1,k]} - i)$. The timeline is:

Common Special Case: If the notional amount is constant for each period (which is the most common situation), then the above expression for the swap rate i will simplify nicely, as follows:

Since $N_0 = N_1 = \dots$, they all cancel out in the above expression for i , giving

$$i = \frac{f_{[0,1]} \cdot (v_1) + f_{[1,2]} \cdot (v_2)^2 + \dots}{(v_1) + (v_2)^2 + \dots}$$

The denominator will not simplify, but the numerator will. Each term in the numerator looks like $f_{[k-1,k]} \cdot (v_k)^k$ and since $f_{[k-1,k]} = \frac{(1+s_k)^k}{(1+s_{k-1})^{k-1}} - 1 = \frac{(v_k)^{-k}}{(v_{k-1})^{-(k-1)}} - 1$, we get

$$f_{[k-1,k]} \cdot (v_k)^k = \left(\frac{(v_k)^{-k}}{(v_{k-1})^{-(k-1)}} - 1 \right) \cdot (v_k)^k = (v_{k-1})^{(k-1)} - (v_k)^k$$

The punchline here is the last simplification; namely that

$$\boxed{f_{[k-1,k]} \cdot (v_k)^k = (v_{k-1})^{(k-1)} - (v_k)^k} \quad (\text{this is a very "useful formula"})$$

Going back to the numerator in the expression for i , the terms become

$$\begin{aligned} f_{[0,1]} \cdot (v_1) &= (v_0)^0 - (v_1)^1 = 1 - v_1 \\ f_{[1,2]} \cdot (v_2)^2 &= (v_1)^1 - (v_2)^2 = v_1 - (v_2)^2 \\ f_{[2,3]} \cdot (v_3)^3 &= (v_2)^2 - (v_3)^3 \\ &\vdots \end{aligned}$$

Therefore, the numerator simplifies by being a telescoping sum.

Example: Beth has a variable-rate loan of 1000 for the next four years where the interest rate is reset annually to the one-year spot interest rate. Beth enters into an interest rate swap with a four-year term and annual settlement periods. Under the swap, Beth will pay the fixed rate and receive the variable rate. (Beth is the payer.) Calculate the fixed interest rate using the spot rates at the beginning of this document. Also determine the net swap payment for the second year.

Solution: The wording of the problem is just describing an interest rate swap with a constant notional amount of 1000. Since the notional amount is constant, the value of the swap rate does not depend on the notional amount. With a 4-year term, we get

$$i = \frac{f_{[0,1]} \cdot (v_1) + f_{[1,2]} \cdot (v_2)^2 + f_{[2,3]} \cdot (v_3)^3 + f_{[3,4]} \cdot (v_4)^4}{(v_1) + (v_2)^2 + (v_3)^3 + (v_4)^4}$$

Using the “useful formula” in the box above for each term in the numerator, we get

$$i = \frac{[1 - (v_1)] + [(v_1) - (v_2)^2] + [(v_2)^2 - (v_3)^3] + [(v_3)^3 - (v_4)^4]}{(v_1) + (v_2)^2 + (v_3)^3 + (v_4)^4}$$

which telescopes to

$$i = \frac{1 - (v_4)^4}{(v_1) + (v_2)^2 + (v_3)^3 + (v_4)^4}$$

Since $v_1 = \frac{1}{1+s_1} = \frac{1}{1.01}$ and $v_2 = \frac{1}{1+s_2} = \frac{1}{1.02}$ and $v_3 = \frac{1}{1+s_3} = \frac{1}{1.03}$ and $v_4 = \frac{1}{1+s_4} = \frac{1}{1.04}$

we get $i = 0.03901 \dots$. This is the swap rate as an annual effective interest rate.

For year 2, Beth receives from the counterparty $1000 \cdot f_{[1,2]}$ and pays the counterparty $1000 \cdot i$. Since $f_{[1,2]} = \frac{(v_2)^{-2}}{(v_1)^{-1}} - 1 = 0.03009 \dots$, the net swap payment for year 2 is

$$\Delta = 1000 \cdot f_{[1,2]} - 1000 \cdot i = 1000 \cdot (0.03009 \dots - 0.03901 \dots) = -8.92$$

The negative value of Δ indicates that Beth will make a net payment of 8.92 to the counterparty at the end of year 2. This is Beth’s *expected* payment to the counterparty at the end of year 2, since it is based on the spot rates in effect at time 0, when Beth entered into the interest rate swap. Over time, the term structure of interest rates will change. Once we know the *actual* 1-year spot rate at time 1, we can calculate the *actual* net swap payment for year 2. For example, if one year after Beth entered the interest rate swap, the new 1-year spot rate in effect at that time is 4.5%, then the net swap payment for year 2 is $\Delta = 1000 \cdot (0.045) - 1000 \cdot i = 5.98$, indicating that Beth would be *receiving* a net payment 5.98 from the counterparty at the end of year 2.

MARKET VALUE OF SWAP

Taking the logic in the last paragraph a step further, we can calculate the **market value** of the interest rate swap one year after Beth entered the interest rate swap. For example, suppose one year after Beth entered into the interest rate swap, the new term structure of interest rates for the next four years is:

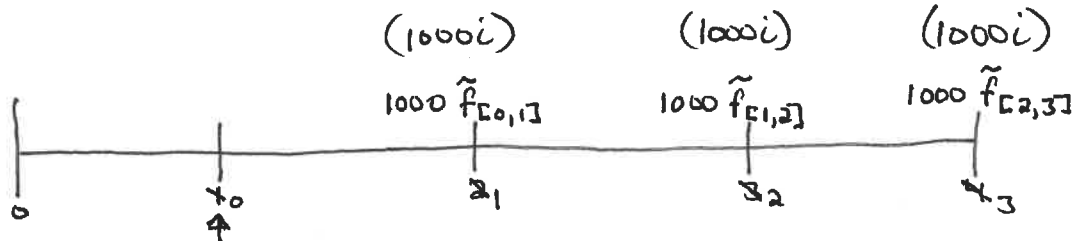
$\tilde{s}_1 = 0.045$ (new 1-year spot rate)

$\tilde{s}_2 = 0.055$ (new 2-year spot rate)

$\tilde{s}_3 = 0.065$ (new 3-year spot rate)

$\tilde{s}_4 = 0.075$ (new 4-year spot rate)

Using this new term structure of interest rates, we will generally get different net swap payments than what was calculated with the original term structure of interest rates. The market value of the swap at this time is the present value of the new net swap payments using the new term structure of interest rates. For example, letting $\tilde{f}_{[k-1,k]}$ denote the new forward rates from time $k - 1$ to time k , we get the timeline



Therefore, the market value of Beth's interest rate swap one year after it was entered is

$$MV_1 = 1000 \cdot [(\tilde{f}_{[0,1]} - i) \cdot (\tilde{v}_1) + (\tilde{f}_{[1,2]} - i) \cdot (\tilde{v}_2)^2 + (\tilde{f}_{[2,3]} - i) \cdot (\tilde{v}_3)^3]$$

where $\tilde{v}_k = \frac{1}{1+\tilde{s}_k}$. At this point, we can just do the calculation, or we can recognize that the expression in the brackets can be rewritten as

$$\tilde{f}_{[0,1]} \cdot (\tilde{v}_1) + \tilde{f}_{[1,2]} \cdot (\tilde{v}_2)^2 + \tilde{f}_{[2,3]} \cdot (\tilde{v}_3)^3 - i \cdot (\tilde{v}_1 + (\tilde{v}_2)^2 + (\tilde{v}_3)^3)$$

Using the "useful formula" in the box above, the first three terms in this last expression will telescope, and we'll get

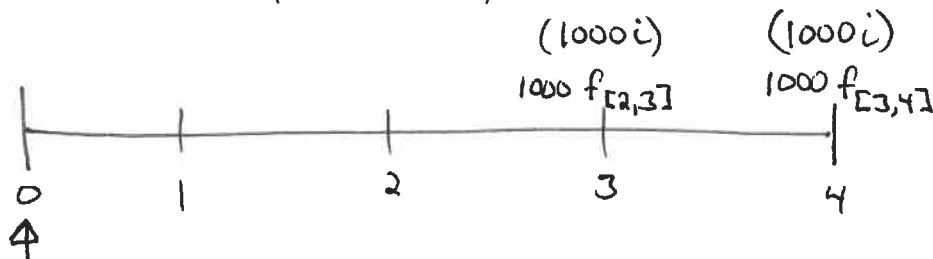
$$MV_1 = 1000 \cdot [(1 - (\tilde{v}_3)^3) - i \cdot (\tilde{v}_1 + (\tilde{v}_2)^2 + (\tilde{v}_3)^3)]$$

Either way, we get $MV_1 = 67.455 \dots$. The positive value indicates Beth's position is worth 67.46. This is how much Beth would receive if she sells the swap at time 1.

Note that the market value of an interest rate swap at the time the swap is entered into is 0. In fact, the swap rate could have been defined as the fixed interest rate that makes the market value of the swap equal 0 at the time the swap is entered into.

DEFERRED SWAP

Suppose Beth's interest rate swap is such that the variable interest rate is swapped for a fixed interest rate for only the last two of the four years. This is referred to as a **deferred** interest rate swap. Again, since the notional amount is constant, we can ignore it in our calculations (it will cancel off). The timeline is:



Solving for i , we get

$$i = \frac{f_{[2,3]} \cdot (v_3)^3 + f_{[3,4]} \cdot (v_4)^4}{(v_3)^3 + (v_4)^4}$$

Once again, using the “useful formula” in the above box, the numerator telescopes and we get

$$i = \frac{(v_2)^2 - (v_4)^4}{(v_3)^3 + (v_4)^4}$$

This i is called the 2-year deferred 2-year interest rate swap rate.

Finally, note that our interest rate swap examples have been ones with a constant notional amount, since the expression for the swap rate telescopes to give an expression that is easier to calculate. If the notional amounts are not constant, then we have no other choice than to just use brute force as the timeline would indicate. You may still find the “useful formula” useful, but the expression for the swap rate will not simplify as nicely.

In the case that the notional amounts increase over time, we get what’s called an **accreting** swap, whereas if the notional amounts decrease over time, we get what’s called an **amortizing** swap.